

Eigenvalues and extremal degrees in graphs

Vladimir Nikiforov

Department of Mathematical Sciences, University of Memphis,
Memphis TN 38152, USA, email: vnikfrv@memphis.edu

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Abstract

Let G be a graph with n vertices, $\mu_1(G) \geq \dots \geq \mu_n(G)$ be the eigenvalues of its adjacency matrix, and $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G)$ be the eigenvalues of its Laplacian. We show that

$$\delta(G) \leq \mu_k(G) + \lambda_k(G) \leq \Delta(G) \quad \text{for all } 1 \leq k \leq n,$$

and

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq \delta(G) - \Delta(G) - 1 \quad \text{for all } 2 \leq k \leq n.$$

Let \mathcal{G} be an infinite family of graphs. We prove that \mathcal{G} is quasi-random if and only if $\mu_n(G) + \mu_n(\overline{G}) = o(n)$ for every $G \in \mathcal{G}$ of order n . This also implies that if $\lambda_n(G) + \lambda_n(\overline{G}) = n + o(n)$ for every $G \in \mathcal{G}$ of order n , then \mathcal{G} is quasi-random.

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1 Introduction

Our notation is standard (e.g., see [1], [2], and [5]); in particular, all graphs are defined on the vertex set $\{1, 2, \dots, n\}$, $G(n)$ stands for a graph of order n , and \overline{G} denotes the complement of G . Writing $A(G)$ for the adjacency matrix of G and $D(G)$ for the diagonal matrix of its degree sequence, the Laplacian of G is defined as $L(G) = D(G) - A(G)$. If $G = G(n)$, we order the eigenvalues of $A(G)$ as $\mu_1(G) \geq \dots \geq \mu_n(G)$ and the eigenvalues of $L(G)$ as $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G)$.

In this note we prove that if $G = G(n)$ is a graph with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, then

$$\delta(G) \leq \mu_k(G) + \lambda_k(G) \leq \Delta(G) \quad \text{for all } 1 \leq k \leq n. \tag{1}$$

This, in turn, implies that

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq \delta(G) - \Delta(G) - 1 \quad \text{for all } 2 \leq k \leq n, \tag{2}$$

complementing the well-known inequality $\mu_k(G) + \mu_{n-k+2}(\overline{G}) \leq -1$.

In the second part of this note we give new spectral conditions for quasi-randomness of graphs. Throughout this note we denote by \mathcal{G} an infinite family of graphs. Following Chung, Graham, and Wilson [3], we call a family \mathcal{G} *quasi-random*, if for every $G \in \mathcal{G}$ of order n ,

$$\mu_1(G) = 2e(G)/n + o(n), \quad \mu_2(G) = o(n), \quad \text{and} \quad \mu_n(G) = o(n).$$

Applying results of [6], we first prove the following theorem.

Theorem 1 *A family \mathcal{G} is quasi-random if and only if*

$$\mu_n(G) + \mu_n(\overline{G}) = o(n) \tag{3}$$

for every graph $G \in \mathcal{G}$ of order n .

This, in turn, implies the following sufficient conditions for quasi-randomness in terms of Laplacian eigenvalues.

Theorem 2 *If \mathcal{G} is a family such that*

$$\lambda_n(G) + \lambda_n(\overline{G}) = n + o(n) \tag{4}$$

for every $G \in \mathcal{G}$ of order n , then \mathcal{G} is quasi-random.

Since $\lambda_2(G) + \lambda_n(\overline{G}) = n$ for every $G = G(n)$, we also obtain the following theorem.

Theorem 3 *If \mathcal{G} is a family such that*

$$\lambda_2(G) + \lambda_2(\overline{G}) = o(n)$$

for every $G \in \mathcal{G}$ of order n , then \mathcal{G} is quasi-random.

We leave the extension of the above results to normalized Laplacians to the interested reader.

2 Proofs

Proof of inequality (1) Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be orthogonal unit eigenvectors to $\lambda_1, \dots, \lambda_n$. For every $k = 2, \dots, n$, the variational characterization of eigenvalues of Hermitian matrices ([5], p. 178-179) implies that

$$\lambda_k(G) = \min_{\|\mathbf{x}\|=1, \mathbf{x} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}} \langle L\mathbf{x}, \mathbf{x} \rangle \tag{5}$$

$$\mu_k(G) = \min_{M \subset \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp M} \langle A\mathbf{x}, \mathbf{x} \rangle \right\} \tag{6}$$

Let \mathbf{y} be such that $\langle A\mathbf{y}, \mathbf{y} \rangle$ is maximal subject to $\|\mathbf{y}\| = 1$ and $\mathbf{y} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$. Letting $\mathbf{y} = (y_1, \dots, y_n)$, we find that

$$\begin{aligned} \lambda_k(G) &\leq \langle L\mathbf{y}, \mathbf{y} \rangle = \sum_{u \in V(G)} d(u) y_u^2 - \langle A\mathbf{y}, \mathbf{y} \rangle \leq \Delta(G) - \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}} \langle A\mathbf{x}, \mathbf{x} \rangle \\ &\leq \Delta(G) - \min_{M \subset \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp M} \langle A\mathbf{x}, \mathbf{x} \rangle \right\} = \Delta(G) - \mu_k(G), \end{aligned}$$

proving the second inequality of (1). The first inequality is deduced likewise using the dual version of (5) and (6). \square

Proof of inequality (2) It is known that $\lambda_k(G) + \lambda_{n-k+2}(\overline{G}) = n$ for all $2 \leq k \leq n$. This, in view of (1), implies that

$$\begin{aligned} n + \mu_k(G) + \mu_{n-k+2}(\overline{G}) &= \lambda_k(G) + \lambda_{n-k+2}(\overline{G}) + \mu_k(G) + \mu_{n-k+2}(\overline{G}) \\ &\geq \delta(G) + \delta(\overline{G}) \geq \delta(G) + n - 1 - \Delta(G), \end{aligned}$$

completing the proof of (2). \square

Proof of Theorem 1 The necessity of condition (3) is a routine fact, so we shall prove only its sufficiency. Let $G = G(n)$, $e(G) = m$, and set $s(G) = \sum_{u \in V(G)} |d(u) - 2m/n|$. The following results were obtained in [6]

$$\frac{s^2(G)}{2n^2\sqrt{2m}} \leq \mu_1(G) - 2m/n \leq \sqrt{s(G)}, \quad (7)$$

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq -1 - 2\sqrt{2s(G)} \quad \text{for all } 2 \leq k \leq n, \quad (8)$$

$$\mu_n(G) + \mu_n(\overline{G}) \leq -1 - s^2(G) / (2n^3). \quad (9)$$

Hence, if (3) holds, (9) implies $\mu_n(G) = o(n)$, $\mu_n(\overline{G}) = o(n)$, and $s(G) = o(n^2)$. Thus, from (7) we obtain $\mu_1(G) = 2m/n + o(n)$. Since $\mu_2(G) + \mu_n(\overline{G}) \leq -1$, inequality (8) implies that $\mu_2(G) = o(n)$, completing the proof. \square

Proof of Theorem 2 According to Grone and Merris [4], $\lambda_k(G) \geq \Delta(G)$. Thus, (4) implies

$$n - 1 + \Delta(G) - \delta(G) = \Delta(G) + \Delta(\overline{G}) \leq \lambda_n(G) + \lambda_n(\overline{G}) = n + o(n).$$

Hence,

$$\Delta(G) - \delta(G) = \Delta(\overline{G}) - \delta(\overline{G}) = o(n)$$

and (1) implies

$$\begin{aligned} \mu_n(G) &= -\lambda_n(G) + \Delta(G) + o(n) \\ \mu_n(\overline{G}) &= -\lambda_n(\overline{G}) + \delta(\overline{G}) + o(n). \end{aligned}$$

Adding these two inequalities, in view of (4), we obtain $\mu_n(G) + \mu_n(\overline{G}) = o(n)$; the assertion follows from Theorem 1. \square

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